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A CHARACTERIZATION OF THE POWER FUNCTION DISTRIBUTION BY INDEPENDENT PROPERTY OF LOWER RECORD VALUES

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Abstract. We prove a characterization of the power function distribution by lower record values. We prove that $F(x) = (\frac{x}{a})^{\alpha}$ for all x, $0 < x < a$, $\alpha > 0$ and $a > 0$ if and only if $\dfrac{X_{L(n)}}{X_{L(m)}}$ and $X_{L(m)}$ are independent for $1 \leq m < n$.

1. Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed(i.i.d.) random variables with cumulative distribution function(cdf) $F(x)$ and probability density function(pdf) $f(x)$. Let $Y_n = max(min)\{ X_1, X_2, \cdots, X_n\}$ for $n \geq 1$. We say X_j is an upper(lower) record value of this sequence, if $Y_j > (<) Y_{j-1}$ for $j > 1$. We denote by $X_{U(m)}$ and $X_{L(m)}$ the m-th upper and lower record values, respectively. By definition, X_1 is an upper as well as a lower record value. The indices at which the upper record values occur are given by the record times $\{U(n), n \geq 1\}$, where $U(n) = min\{j \mid j > U(n-1), X_j > X_{U(n-1)}, n \geq 2\}$ with $U(1) = 1$. We assume that all upper record values $X_{U(i)}$ for $i \geq 1$ occur at a sequence $\{X_n, n \geq 1\}$ of i.i.d. random variables.

We define the power function distribution of a random variable.

A continuous random variable X is called the power function distribution with parameters $a > 0$, $\alpha > 0$ if its cdf is given by

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$$F(x) = \begin{cases} \left(\dfrac{x}{a}\right)^{\alpha}, & 0 < x < a, \\ 0, & otherwise. \end{cases} \tag{1.1}$$

Ahsanullah(1995) prove that X has exponential distribution if and only if $X_{U(n)} - X_{U(m)}$ and $X_{U(m)}$ are independent. Also, Lee and Lim(2010) show that X has weibull distribution if and only if $\frac{X_{U(m)}}{X_{U(n)}}$ and $X_{U(n)}$ are independent.

In this paper, we obtain a characterization of the power function distribution by independent property of lower record values.

2. Main results

To prove Theorem 2.2, we need the following Lemma 2.1.

LEMMA 2.1. *Let $F(x)$ be an absolutely continuous function and $F(x) > 0$ for all $x > 0$. Suppose that $\frac{F(vw)}{F(v)} = e^{-q(v,w)}$ and $h(v,w)= \{q(v,w)\}^r e^{-q(v,w)}\{-\frac{\partial}{\partial w}q(v,w)\}$ for $r \geq 0$, $h(v,w) \neq 0$, $\frac{\partial}{\partial w}q(v,w) \neq 0$ for any v and w. If $h(v,w)$ is independent of v, then $q(v,w)$ is a function of w only.*

Proof. Let

$$\begin{aligned} g(w) = h(v,w) &= \{q(v,w)\}^r e^{-q(v,w)}\{-\frac{\partial}{\partial w}q(v,w)\} \\ &= \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} q(v,w)^{j+r}\{-\frac{\partial}{\partial w}q(v,w)\}. \end{aligned} \tag{2.1}$$

Integrating (2.1) with respect to w, we obtain

$$\int g(w)dw + c = \sum_{j=0}^{\infty} \frac{(-1)^{j+1}}{j!} q(v,w)^{j+r+1} \frac{1}{(j+r+1)} = G_1(w). \tag{2.2}$$

Here G_1 is a function of w only and c is independent of w but may depend on v.

Now letting $w \to 1$, $q(v,w) \to 0$, we have c independently of v from (2.2). Therefore

$$0 = \frac{\partial}{\partial v} G_1(w) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} q(v,w)^{j+r} \{-\frac{\partial}{\partial v} q(v,w)\}$$
$$= g(w)\{-\frac{\partial}{\partial w} q(v,w)\}^{-1} \{\frac{\partial}{\partial v} q(v,w)\}.$$

We know $h(v,w) \neq 0$ and $\frac{\partial}{\partial w} q(v,w) \neq 0$, so we must have

$$\frac{\partial}{\partial v} q(v,w) = 0.$$

Hence $q(v,w)$ is a function of w only. □

THEOREM 2.2. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with cdf $F(x)$ which is absolutely continuous with pdf $f(x)$ and $F(a) = 1$ and $F(x) < 1$ for all x, $0 < x < a$. Then $F(x) = (\frac{x}{a})^\alpha$ for all x, $0 < x < a$ and $a > 0$ if and only if $\frac{X_{L(n)}}{X_{L(m)}}$ and $X_{L(m)}$ are independent for $1 \leq m < n$.*

Proof. The joint pdf $f_{m,n}(x,y)$ of $X_{L(m)}$ and $X_{L(n)}$ is found to be

$$f_{m,n}(x,y) = \frac{\{H(x)\}^{m-1}}{(m-1)!} h(x) \frac{\{H(y) - H(x)\}^{n-m-1}}{(n-m-1)!} f(y),$$

where $H(x) = -lnF(x)$ and $h(x) = -\frac{d}{dx} H(x)$.

Consider the functions $V = X_{L(m)}$ and $W = \frac{X_{L(n)}}{X_{L(m)}}$. It follows that $x_{L(m)} = v$, $x_{L(n)} = vw$ and $\mid J \mid = v$. Thus we can find the joint pdf $f_{V,W}(v,w)$ of V and W as

$$f_{V,W}(v,w) = \frac{\{H(v)\}^{m-1}}{(m-1)!} h(v) \frac{\{H(vw) - H(v)\}^{n-m-1}}{(n-m-1)!} f(vw)v$$

for $0 < v < a$, $0 < w < 1$.

If $F(x) = (\frac{x}{a})^\alpha$ for all $0 < x < a$ and $a > 0$, then we get

$$\begin{aligned} f_{V,W}(v,w) &= \frac{\alpha^2}{a(m-1)!(n-m-1)!} \\ &\times \{-\alpha \ln(\frac{v}{a})\}^{m-1} \{-\alpha \ln w\}^{n-m-1} (\frac{v}{a})^{\alpha-1} w^{\alpha-1} \end{aligned} \tag{2.3}$$

for all $v > 1$, $w > 1$ and $\alpha > 0$.

The marginal pdf of W is given by

$$f_W(w) = \int_0^a f_{V,W}(v,w)dv = \alpha\frac{\{-\alpha\ln w\}^{n-m-1}}{(n-m-1)!}w^{\alpha-1} \tag{2.4}$$

for all $0 < w < 1$, $\alpha > 0$.

Also, the pdf $f_V(v)$ is given by

$$\begin{aligned} f_V(v) &= \frac{\{H(v)\}^{n-2}}{(n-2)!}f(v) \\ &= \frac{\alpha}{a(m-1)!}\{-\alpha\ln(\frac{v}{a})\}^{m-1}(\frac{v}{a})^{\alpha-1}. \end{aligned} \tag{2.5}$$

From (2.3), (2.4), and (2.5), we obtain $f_{V,W}(v,w) = f_V(v)f_W(w)$. Hence V and W are independent for $1 < m < n$.

Now we will prove the sufficient condition. Let us use the transformation $V = X_{L(m)}$ and $W = \dfrac{X_{L(n)}}{X_{L(m)}}$. The Jacobian of the transformation is $\mid J \mid= v$. Thus we can find the joint pdf $f_{V,W}(v,w)$ of V and W as

$$f_{V,W}(v,w) = \frac{\{H(v)\}^{m-1}}{(m-1)!}h(v)\frac{\{H(vw)-H(v)\}^{n-m-1}}{(n-m-1)!}f(vw)v \tag{2.6}$$

for all $0 < v < a$, $0 < w < 1$ and $\alpha > 0$.

The pdf $f_V(v)$ is given by

$$f_V(v) = \frac{\{H(v)\}^{m-1}}{(m-1)}f(v) \tag{2.7}$$

for all $0 < v < a$, $m > 1$.

From (2.6) and (2.7), we can get the conditional pdf of $f_W(w|v)$ as

$$\begin{aligned} &f_W(w|X_{L(m)} = v) \\ &= \frac{\{H(vw)-H(v)\}^{n-m-1}}{(n-m-1)!}\frac{f(vw)v}{F(v)} \\ &= \frac{1}{(n-m-1)!}\left(-\ln\frac{F(vw)}{F(v)}\right)^{n-m-1}\frac{f(vw)v}{F(v)} \\ &= \frac{1}{(n-m-1)!}\left(-\ln\frac{F(vw)}{F(v)}\right)^{n-m-1}\left(\frac{F(vw)}{F(v)}\right)\left(-\frac{\partial}{\partial w}\left(-\ln\frac{F(vw)}{F(v)}\right)\right). \end{aligned}$$

Since V and W are independent, by using Lemma 2.1, $q(v,w) = -\ln\frac{F(vw)}{F(v)}$ is a function of w only. Thus

$$\frac{F(vw)}{F(v)} = G(w),$$

where $G(w)$ is a function of w only. Taking $v \to a-$, we get $G(w) = F(aw)$. Thus

$$F(vw) = F(v)F(aw) \tag{2.8}$$

for all $0 < v < a$ and $0 < w < 1$.

By the theory of functional equations, the only continuous solution of (2.8) with the boundary condition $F(0) = 0$ is

$$F(x) = \left(\frac{x}{a}\right)^{\alpha}$$

for all $0 < x < a$ and $\alpha > 0$.

This completes the proof. □

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